Are operads algebraic theories?

Tom Leinster

University of Glasgow T.Leinster@maths.gla.ac.uk www.maths.gla.ac.uk/~tl

Abstract

I exhibit a pair of non-symmetric operads that, although not themselves isomorphic, induce isomorphic monads. The existence of such a pair implies that if 'algebraic theory' is understood as meaning 'monad', operads cannot be regarded as algebraic theories of a special kind.

Introduction

Operads tend to be thought of as algebraic theories of some kind, with the nth piece P(n) of an operad P thought of as the collection of n-ary operations. This point of view seems to be validated by the fact that any operad has a category of algebras. Nevertheless, it is not clear in principle that the passage from an operad to its category of algebras does not involve a loss of information. The purpose of this note is to show by example that such a loss can indeed occur, in the setting of operads without symmetric group action.

The passage from operads to algebraic theories can be expressed more precisely as follows. I use 'operad' to mean 'non-symmetric operad of sets'. Any operad induces a monad on **Set**, the algebras for which are exactly the algebras for the operad. Any map of operads induces a map between the resulting monads, where by definition a map of monads is a natural transformation preserving multiplication and units in an obvious sense made precise below. This defines a functor

$$(operads) \longrightarrow (monads on Set).$$

But this functor does not reflect isomorphism: in other words, there exist non-isomorphic operads P and P' whose associated monads are isomorphic. This implies, of course, that the categories of algebras for P and P' are isomorphic, so P and P' are 'Morita equivalent' in a strong sense. It also implies that an operad should not be regarded as merely a monad with certain properties: the canonical map from isomorphism classes of operads to isomorphism classes of monads is not injective.

Such a pair of operads is constructed as follows. Any operad P gives rise to a new operad P^{rev} , whose induced monad is isomorphic to that of P (Section 1).

It is then just a matter of finding an operad P such that $P \not\cong P^{\text{rev}}$. This is done in Section 2; further comments follow in Section 3.

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1 The reverse of an operad

For each operad P, I define its 'reverse' P^{rev} and show that the monads induced by P and P^{rev} are isomorphic.

Let P be an operad. Its **reverse** P^{rev} is defined as follows: $P^{\text{rev}}(n) = P(n)$ for all $n \in \mathbb{N}$, the identity of P^{rev} is the same as that of P, and the composition \circ_{rev} is given by

$$\theta \circ_{\text{rev}}(\theta_1, \dots, \theta_n) = \theta \circ (\theta_n, \dots, \theta_1)$$

 $(n, k_i \in \mathbb{N}, \theta \in P(n), \theta_i \in P(k_i))$. This does define an operad P^{rev} : all that needs checking is associativity, which is straightforward.

Let (S, μ, η) be the monad on **Set** induced by P. Then for any set X,

$$SX = \sum_{n \in \mathbb{N}} P(n) \times X^n,$$

the unit map

$$\eta_X: X \longrightarrow SX$$

picks out the identity element of P(1), and the multiplication map

$$\mu_X: S^2X \longrightarrow SX$$

is given by

$$\left(\theta, (\theta_1, x_1^1, \dots, x_1^{k_1}), \dots, (\theta_n, x_n^1, \dots, x_n^{k_n})\right) \\ \longmapsto \left(\theta \circ (\theta_1, \dots, \theta_n), x_1^1, \dots, x_1^{k_1}, \dots, x_n^1, \dots, x_n^{k_n}\right)$$

 $(n, k_i \in \mathbb{N}, \theta \in P(n), \theta_i \in P(k_i), x_i^j \in X)$. The monad $(S^{\text{rev}}, \mu^{\text{rev}}, \eta^{\text{rev}})$ induced by P^{rev} is the same except that the multiplication formula becomes

$$\left(\theta, (\theta_1, x_1^1, \dots, x_1^{k_1}), \dots, (\theta_n, x_n^1, \dots, x_n^{k_n})\right) \\ \longmapsto \left(\theta \circ (\theta_n, \dots, \theta_1), x_1^1, \dots, x_1^{k_1}, \dots, x_n^1, \dots, x_n^{k_n}\right).$$

There is a natural isomorphism $\iota: S \xrightarrow{\sim} S^{\text{rev}}$ whose component at a set X is

$$\iota_X: SX \xrightarrow{\sim} S^{\text{rev}}X$$

$$(\theta, x_1, \dots, x_n) \longmapsto (\theta, x_n, \dots, x_1)$$

 $(n \in \mathbb{N}, \theta \in P(n), x_i \in X)$. Using the above descriptions of the monad structures, it is straightforward to check that ι is an isomorphism of monads, in other words, that for each X the diagrams

$$X = X \qquad S^{2}X \xrightarrow{S\iota_{X}} SS^{\text{rev}}X \xrightarrow{\iota_{S^{\text{rev}}X}} (S^{\text{rev}})^{2}X$$

$$\downarrow^{\eta_{X}} \qquad \downarrow^{\eta_{X}^{\text{rev}}} \qquad \mu_{X} \qquad \downarrow^{\mu_{X}^{\text{rev}}}$$

$$SX \xrightarrow{\iota_{X}} S^{\text{rev}}X \qquad SX \xrightarrow{\iota_{X}} S^{\text{rev}}X$$

commute.

So, as promised, any operad P gives rise to a new operad P^{rev} inducing the same monad as P.

There are at least two abstract perspectives on this construction. First, write T for the free monoid monad on **Set**. Then an operad amounts to a cartesian monad $S = (S, \mu, \eta)$ on **Set** together with a cartesian natural transformation $\pi: S \longrightarrow T$ respecting the monad structures, and the monad induced by the operad is simply S. (For explanation and proof, see for instance Cor 6.2.4 of [L].) Now, there is an involution ρ of the monad T given by reversing the order of finite lists, which implies that any operad P described by a pair (S, π) gives rise to a new operad described by the pair $(S, \rho \circ \pi)$; this operad is P^{rev} . From this point of view, the monad induced by P^{rev} is not just isomorphic but equal to that induced by P.

Second, given any cartesian monad T on a category \mathcal{E} with pullbacks, there is a category of so-called T-operads. (See Chapter 4 of [L].) Any T-operad P induces a monad T_P on \mathcal{E} , and algebras for the operad are by definition algebras for this monad. When T is the free monoid monad on \mathbf{Set} , these are the standard notions of non-symmetric operad, induced monad, and algebra. Inevitably, if we have an isomorphism $(\mathcal{E},T) \stackrel{\sim}{\longrightarrow} (\mathcal{E}',T')$ between two different cartesian monads on two different categories then there is an induced isomorphism between the categories of T-operads and T'-operads, and if P' is the T'-operad corresponding to a T-operad P then the monad $T_{P'}$ on \mathcal{E}' is obtained by transporting the monad T_P on \mathcal{E} across the isomorphism. In particular, this holds for the isomorphism

$$(\mathrm{id}, \rho) : (\mathbf{Set}, T) \xrightarrow{\sim} (\mathbf{Set}, T)$$

where T is the free monoid monad and ρ is as above; the resulting automorphism of the category of non-symmetric operads sends P to $P^{\rm rev}$, and by the preceding comments the respective induced monads are isomorphic.

Observe also that reversal works for (non-symmetric) operads in any symmetric monoidal category \mathcal{V} . The definition of P^{rev} is an absolutely straightforward generalization of the case $\mathcal{V} = \mathbf{Set}$, using the symmetry of \mathcal{V} . If \mathcal{V} has countable coproducts and tensor distributes over them then any operad P in \mathcal{V} induces a monad on \mathcal{V} , algebras for which are algebras for P; and just as above, the monads induced by P and P^{rev} are isomorphic.

2 The counterexample

To find a pair of non-isomorphic operads whose induced monads are isomorphic, it suffices to find an operad not isomorphic to its reverse.

This task is not completely straightforward, since many commonly encountered operads admit a symmetric structure and any such operad is isomorphic to its reverse. Indeed, let $\sigma_n \in S_n$ denote the permutation reversing the order of n letters: then for any symmetric operad P, there is an isomorphism $P \xrightarrow{\sim} P^{\text{rev}}$ sending $\theta \in P(n)$ to $\theta \cdot \sigma_n \in P^{\text{rev}}(n)$. Further, several well-known operads that do not admit a symmetric structure are, nevertheless, isomorphic to their reverse: this applies, for instance, to Stasheff's operad of associahedra ([S], [MSS]).

Here is an operad P not isomorphic to its reverse. Let P(n) be the set of all n-tuples (f_1, \ldots, f_n) of order-preserving continuous maps $f_i : [0,1) \longrightarrow [0,1)$ of the half-open real interval such that if i < j and $t_i, t_j \in [0,1)$ then $f_i(t_i) < f_j(t_j)$. The identity of P is the identity map $\mathrm{id}_{[0,1)} \in P(1)$, and composition

$$P(n) \times P(k_1) \times \cdots \times P(k_n) \longrightarrow P(k_1 + \cdots + k_n)$$

is

$$((f_1, \dots, f_n), (f_1^1, \dots, f_1^{k_1}), \dots, (f_n^1, \dots, f_n^{k_n})) \\ \longmapsto (f_1 f_1^1, \dots, f_1 f_1^{k_1}, \dots, f_n f_n^1, \dots, f_n f_n^{k_n}).$$

Seen another way, P is an endomorphism operad. For consider the (non-symmetric) monoidal category of ordered topological spaces, where the product $X \boxplus Y$ is defined by taking the disjoint union of X and Y and adjoining the relation x < y for each $x \in X$ and $y \in Y$. (Compare addition of ordinals.) Then P(n) is the set of maps $[0,1)^{\boxplus n} \longrightarrow [0,1)$, with the usual endomorphism operad structure.

To prove that P is not isomorphic to P^{rev} , I introduce some temporary terminology. Let Q be an operad. An element $\gamma \in Q(1)$ is **constant** if

for all
$$n \in \mathbb{N}$$
 and all $\phi, \phi' \in Q(n), \gamma \circ (\phi) = \gamma \circ (\phi')$.

An element $\phi \in Q(n)$ is surjective if

for all
$$\theta, \theta' \in Q(1)$$
, $\theta \circ (\phi) = \theta' \circ (\phi) \Rightarrow \theta = \theta'$.

The following lemma shows that these terms have the expected meanings when Q is P or P^{rev} . For convenience, I write an element $(g) \in P(1)$ as simply g.

Lemma

a. $g \in P(1)$ is constant in the sense above if and only if the map $g : [0,1) \longrightarrow [0,1)$ is constant in the usual sense.

b. $(f_1, \ldots, f_n) \in P(n)$ is surjective in the sense above if and only if the union of the images of f_1, \ldots, f_n is [0, 1).

Moreover, both statements remain true when P is replaced by P^{rev} .

Proof For (a), 'if' is clear. Now suppose g is not constant in the usual sense, so that there exist $t, t' \in [0, 1)$ with $g(t) \neq g(t')$. If we take $f, f' : [0, 1) \longrightarrow [0, 1)$ to be the constant functions with respective values t and t' then $f, f' \in P(1)$ with $g \circ (f) \neq g \circ (f')$, so g is not constant in the sense above.

For (b), 'if' is also clear. Conversely, if the union of the images of f_1, \ldots, f_n is not [0, 1) then by continuity, one of the following holds:

- n = 0
- $n \ge 1$ and $f_1(0) > 0$
- $\sup f_{i-1} < f_i(0)$ for some $i \in \{2, ..., n\}$
- $n \ge 1$ and $\sup f_n < 1$.

In all cases, there is some nonempty open interval $(a, b) \subseteq [0, 1)$ that does not meet the union of the images of f_1, \ldots, f_n . We can construct a continuous order-preserving map $h:[0,1) \longrightarrow [0,1)$ that is not the identity but satisfies h(t) = t for all $t \notin (a, b)$, and this gives distinct elements h, id of P(1) satisfying $h \circ (f_1, \ldots, f_n) = \mathrm{id} \circ (f_1, \ldots, f_n)$. So (f_1, \ldots, f_n) is not surjective in the sense above.

'Moreover' follows immediately from the definition of P^{rev} .

We can now show that the following isomorphism-invariant property of an operad Q holds when Q = P but fails when $Q = P^{\text{rev}}$:

there exist $\phi \in Q(2)$ and constant $\gamma \in Q(1)$ such that $\phi \circ (\gamma, id) \in Q(2)$ is surjective.

It will follow that $P \ncong P^{\text{rev}}$.

To see that the property holds for Q = P, let $g, f_1 : [0, 1) \longrightarrow [0, 1)$ both be the map with constant value 0, and let $f_2 : [0, 1) \longrightarrow [0, 1)$ be the identity. Then $g \in P(1)$ is constant, $\phi = (f_1, f_2)$ is an element of P(2), and if $e = \phi \circ (g, id)$ then

$$e_2 = f_2 \circ \mathrm{id} = \mathrm{id}$$

so e is surjective.

To see that the property fails for $Q = P^{\text{rev}}$, we have to see that given $\phi = (f_1, f_2) \in P(2)$ and constant $g \in P(1)$, the composite $e = \phi \circ (\text{id}, g)$ in P cannot be surjective. Indeed, let b be the constant value of g: then

$$e_2 = f_2 \circ g = (\text{constant map with value } f_2(b)),$$

so by order-preservation

$$image(e_1) \cup image(e_2) \subseteq [0, f_2(b)] \subseteq [0, 1),$$

as required.

3 Further comments

The properties of the functor

$$G: (operads) \longrightarrow (monads on Set)$$

can be analyzed more precisely. The monads in the essential image of G (that is, the monads isomorphic to G(P) for some operad P) are the strongly regular finitary monads. By definition, this is the class of monads whose corresponding algebraic theory can be presented by finitary operations and equations in which the same variables appear on each side of the equals sign, in the same order and without repetition. (For instance, the theory of monoids is allowed, but the theories of commutative monoids and groups are not.) Another description is that they are the cartesian monads S such that there exists a cartesian natural transformation, respecting the monad structures, from S to the free monoid monad. The original source on strong regularity is [CJ]; proofs of the results just mentioned can be found in C.1 and 6.2 of [L].

The functor G does not reflect isomorphism, as has been shown. It does reflect isomorphisms (plural): that is, if $f:P\longrightarrow P'$ is a map of operads and G(f) is an isomorphism then so too is f. This is easily shown, as is the fact that G is faithful. But since a full and faithful functor reflects isomorphism, G cannot be full. To prove this more directly, let P be the functor of the previous section and take the isomorphism $\iota:S\stackrel{\sim}{\longrightarrow} S^{\text{rev}}$ of Section 1, where S and S^{rev} are the monads induced by P and P^{rev} respectively. Then since G reflects isomorphisms, there is no map $f:P\longrightarrow P^{\text{rev}}$ satisfying $G(f)=\iota$: so again, G is not full.

Here I have stuck to operads of sets; I know little about the situation for other types of operad. Trivially, taking discrete spaces on the set-theoretic example above yields a pair of non-symmetric topological operads that induce isomorphic monads but are not themselves isomorphic.

The situation for symmetric operads is completely different: symmetric operads of sets can be identified as monads of a special kind. Precisely, the canonical functor

(symmetric operads)
$$\longrightarrow$$
 (monads on **Set**)

defines an equivalence between the category of symmetric operads and the category of analytic monads and weakly cartesian maps. This is a result of Weber [W], using Joyal's characterization of the endofunctors on **Set** induced by species and of the natural transformations induced by maps between them [J].

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